QR-Type Factorizations, the Yang-Baxter Equation, and an Eigenvalue Problem of Control Theory

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ABSTRACT

It is shown that every solution to the Yang-Baxter equation corresponds with the *QR*-type algorithm. The structure of solutions of the Yang-Baxter equation is described and examples relevant to the algebraic Riccati equation are considered.

1. INTRODUCTION

In the present paper we consider a group-theoretic setting for the *QR*-type algorithms and related dynamical systems. There has been considerable interest in this topic in the past few years. The specialists in mathematical physics [7, 10, 18], dynamical system theory [17], system theory [1, 9], and of course numerical analysis [6, 13, 20, 21] have made their contributions. It is a really interdisciplinary topic. Therefore it is no wonder that it was R. Hermann [11] who proposed a group-theoretic setting for *QR*-type algorithms for the first time. He also indicated an intimate relationship between the topic under consideration and Riccati differential equations.

As is well known, *QR*-type algorithms are associated with certain factorizations in Lie groups (see e.g. [20]). In this paper we address the following question: What are the most general such factorizations? It appears that we can associate a *QR*-type algorithm with every solution to the Yang-Baxter equation. We also describe relevant dynamical systems. We consider two symplectic *QR*-type algorithms for the solution of the algebraic Riccati equation, thus establishing the linkage between the topic under consideration and control theory. Our account is self-contained and can be of an interest for nonspecialists in the field (e.g. for system theorists). Almost all that we need from Lie group theory can be found in the introductory text [19].

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2. GROUP-THEORETIC SCHEME FOR QR-TYPE ALGORITHMS

Let G be a Lie group, G_- a Lie subgroup of G. Consider the following problem [11]. Given $g \in G$, find $q \in G$ so that

$$q \cdot g \cdot q^{-1} \in G_- . \tag{2.1}$$

Let, further, $P_i : G \to G$, $i = 0, 1, \ldots$, be functions defined at least on the conjugacy class $C(g) \triangleq \{\bar{g}g\bar{g}^{-1} : \bar{g} \in G\}$, and such that

$$P_i(\bar{g}g\bar{g}^{-1}) = \bar{g}P_i(g)\bar{g}^{-1}, \qquad \bar{g} \in G. \tag{2.2}$$

We introduce the following functions $\xi_i : G \to G$ ("generalized power iterations"):

$$\xi_i(\bar{g}) = P_i(\bar{g}) \cdot P_{i-1}(\bar{g}) \cdot \cdots \cdot P_0(\bar{g}), \qquad \bar{g} \in C(g), \quad i = 0, 1, \ldots .$$

One of the approaches to the problem (2.1) is to consider the asymptotic behaviour of the sequence $\xi_i(g) \in G_-$ on the left coset space G/G_- endowed with the factor topology [19].

PROPOSITION 1. *Let G_- be a closed subgroup of a Lie group G, and assume that there exists a limit in G/G_-,*

$$\lim \xi_i(g) \cdot G_- = qG_-, \qquad i \to +\infty. \tag{2.3}$$

Let, further,

$$\lim \xi_i(g) \cdot gG_- = q \cdot G_-, \qquad i \to +\infty. \tag{2.4}$$

Then $q^{-1} \cdot g \cdot q \in G_-$.

Proof. By (2.3) we have $g \cdot q \cdot G_- = g(\lim \xi_i(g) \cdot G_-) = \lim(g \cdot \xi_i(g) \cdot G_-)$, $i \to +\infty$. By virtue of (2.2), $\xi_i(g) \cdot g = g \cdot \xi_i(g)$, and by (2.4), $\lim[g \cdot \xi_i(g) \cdot G_-] = qG_-$, $i \to +\infty$. In other words, $q^{-1}gqG_- = G_-$, that is, $q^{-1} \cdot g \cdot q \in G_-$. ■

REMARK. If $P_i(g) = g$, $i = 0, 1, \ldots$, we arrive at the usual power iterations. In this case (2.4) follows from (2.3).

Let G_+ be another closed Lie subgroup of G such that the following condition holds:

C1. There exists a neighborhood U of the identity element e in G and smooth functions $\rho_\pm : U \to G_\pm$ such that

$$g = \rho_+(g)\cdot\rho_-(g), \quad g \in U, \qquad \rho_\pm(e) = e. \tag{2.5}$$

By standard topological considerations [19] the mapping $\Psi : U/G_- \to \rho_+(U)/G_+ \cap G_-$ given by

$$\Psi(g\cdot G_-) = \rho_+(g)\cdot(G_+ \cap G_-), \qquad g \in U,$$

is a topological isomorphism.

PROPOSITION 2. *Under the assumptions of Proposition* 1 *there exists a sequence* $h_i \in G_+ \cap G_-$, $i = 0, 1, \dots,$ *such that*

$$\lim\left\{ h_i^{-1}[\rho_+(\xi_i(g))]^{-1} g [\rho_+(\xi_i(g))]\cdot h_i \right\}$$

$$= \rho_+(q)^{-1} g \rho_+(q) \in G_-, \qquad i \to +\infty, \tag{2.6}$$

provided $\xi_i(g) \in U$, $q \in U$, $i = 0, 1, \dots$.

Proof. By Proposition 1 we have $\lim \xi_i(g)\cdot G_- = q\cdot G_-$, $i \to +\infty$, $q^{-1}gq \in G_-$. Applying the isomorphism Ψ, we obtain $\lim[\rho_+(\xi_i(g))\cdot(G_+ \cap G_-)] = \rho_+(q)\cdot(G_+ \cap G_-)$, $i \to +\infty$. Hence (by the properties of factor topology) there exists a sequence $h_i \in G_+ \cap G_-$, $i = 0, 1, 2, \dots,$ such that

$$\lim \rho_+(\xi_i(g))\cdot h_i = \rho_+(q), \qquad i \to +\infty.$$

This implies (2.6). Moreover,

$$\rho_+(q)^{-1}\cdot g\cdot\rho_+(q) = \rho_-(q)[q^{-1}gq]\rho_-(q)^{-1} \in G_-. \qquad \blacksquare$$

REMARK. In practical situations the convergence properties of QR-type algorithms can be deduced from Proposition 2.

Assume $P_0(g) \in U$, $P_0(g) = g_0^+ \cdot g_0^-$, where $g_0^\pm = \rho_\pm(P_0(g))$. Set $g_1 = (g_0^+)^{-1} g g_0^+$. Suppose we have already constructed a sequence $g_0, g_1, \dots, g_k$ and $P_k(g_k) \in U$. Consider the factorization

$$P_k(g_k) = g_k^+ g_k^-, \qquad g_k^\pm = \rho_\pm(P_k(g_k)). \tag{2.7}$$

We set

$$g_{k+1} = (g_k^+)^{-1} g_k g_k^+ = g_k^- g_k (g_k)^{-1}. \tag{2.8}$$

Let, further,

$$Q_i = g_0^+ \cdot g_1^+ \cdots g_i^+, \quad R_i = g_i^- g_{i-1}^- \cdots g_0^-, \qquad i = 0, 1, \dots. \tag{2.9}$$

PROPOSITION 3. *We have*

$$g_{i+1} = Q_i^{-1} g Q_i = R_i g R_i^{-1}, \tag{2.10}$$

$$\xi_i(g) = Q_i R_i, \qquad i = 0, 1, 2, \dots. \tag{2.11}$$

Proof. By virtue of (2.8) we have $g_1 = (g_0^+)^{-1} g g_0^+ = Q_0^{-1} g Q_0 = g_0^- [P_0(g)]^{-1} g P_0(g) (g_0^-)^{-1} = g_0^- g (g_0^-)^{-1} = R_0 g R_0^{-1}$. Here we have used the basic property (2.2). Proceed by induction. If $g_{i+1} = Q_i^{-1} g Q_i = R_i g R_i^{-1}$, $i \leqslant k-1$, then by (2.8),(2.9), $g_{k+1} = (g_k^+)^{-1} g_k g_k^+ = (g_k^+)^{-1} Q_{k-1}^{-1} g Q_{k-1} g_k^+ = Q_k^{-1} g_k^+ Q_k$, and similarly $g_{k+1} = g_k^- g_k (g_k^-)^{-1} = g_k^- R_{k-1} g R_{k-1}^{-1} (g_k^-)^{-1} = R_k g R_k^{-1}$. Further, $Q_0 R_0 = g_0^+ g_0^- = P_0(g_0) = \xi_0(g)$, and by (2.8),(2.9),(2.7), (2.2) we have $Q_{k+1} R_{k+1} = Q_k g_{k+1}^+ g_{k+1}^- R_k = Q_k P_{k+1}(g_{k+1}) R_k = Q_k P_{k+1}(Q_k^{-1} g Q_k) R_k = Q_k Q_k^{-1} P_{k+1}(g) Q_k R_k = P_{k+1}(g) Q_k R_k$. Hence, $Q_k R_k = \xi_k(g)$ *implies* $Q_{k+1} R_{k+1} = \xi_{k+1}(g)$. ∎

REMARK. Clearly, the iterative procedure described in (2.7),(2.8) is an ordinary generalized QR algorithm (see e.g. [20]). The difference is that we don't impose any restrictions on $G_\pm$ (e.g. $G_+ \cap G_- = e$) except for condition C1.

We can conclude from (2.11) that there exists a sequence $\tilde{h}_i \in G_+ \cap G_-$, $i = 0, 1, \dots$, such that

$$\rho_+(\xi_i(g)) = Q_i \tilde{h}_i. \tag{2.12}$$

Consequently we have obtained

COROLLARY 1. *Under the assumptions of Proposition* 1 *there exists a sequence* $\bar{h}_i \in G_+ \cap G_-$, $i = 0,1,\dots,$ *such that*

$$\lim\left(h_i^{-1}Q_i^{-1}gQ_ih_i\right) = \rho_+(q)^{-1}g\rho_+(q) \in G_-, \qquad i \to \infty. \quad (2.13)$$

3. QR FACTORIZATIONS AND THE YANG-BAXTER EQUATION

Proposition 3 and (2.12) motivate the following question: What additional restrictions must be imposed on the factorization (2.5) to guarantee the equalities

$$\rho_+(\xi_i(g)) = Q_i, \quad \rho_-(\xi_i(g)) = R_i, \qquad i = 0,1,\dots? \qquad (3.1)$$

The following condition seems to be a rather natural candidate.

C2. Given $g, h \in U$ such that $g * h \in U$, where by definition

$$g * h \triangleq \rho_+(g)\rho_+(h)\rho_-(h)\rho_-(g), \qquad (3.2)$$

one has

$$\rho_+(g * h) = \rho_+(g)\rho_+(h), \qquad (3.3)$$

and hence

$$\rho_-(g * h) = \rho_-(h)\rho_-(g). \qquad (3.4)$$

PROPOSITION 4. *Let* $U = G$ (*we use this restriction only for simplicity of notation*) *and conditions* C1, C2 *hold. Then a pair* $(G, *)$ [*see* (3.2)] *is a Lie group, and the mapping* $\varphi: (G, *) \to G_+ \times G_-$, $\varphi(g) = (\rho_+(g), \rho_-(g)^{-1})$, *is an injective Lie group homomorphism. Besides*, (3.1) *holds.*

Proof. Clearly $e * h = \rho_+(e)h\rho_-(e) = h$ by (2.5), and $h * e = \rho_+(h)e\rho_-(h) = e$.

Given $g \in G$, let $h = \rho_+(g)^{-1}\rho_-(g)^{-1}$. By (3.1), $g * h = \rho_+(g)h\rho_-(g) = \rho_+(g)\rho_+(g)^{-1}\rho_-(g)^{-1}\rho_-(g) = e$. Hence, by (2.5),(3.3), $\rho_+(e) = e =$

$\rho_+(g*h)=\rho_+(g)\rho_+(h)$, or $\rho_+(h)=\rho_+(g)^{-1}$, and consequently $\rho_-(h)=\rho_-(g)^{-1}$. This implies $h*g=\rho_+(g)^{-1}g\rho_-(g)^{-1}=e$. Further, $\rho_+((h_1*h_2)*h_3)=\rho_+(h_1*h_2)\rho_+(h_3)=\rho_+(h_1)\rho_+(h_2)\rho_+(h_3)$ by (3.3), and similarly $\rho_+(h_1*(h_2*h_3))=\rho_+(h_1)\rho_+(h_2)\rho_+(h_3)$. Making use of (3.4), we obtain $\rho_-((h_1*h_2)*h_3)=\rho_-(h_1*(h_2*h_3))=\rho_-(h_3)\rho_-(h_2)\rho_-(h_1)$. This yields $h_1*(h_2*h_3)=(h_1*h_2)*h_3$. In particular, by virtue of (2.11),(2.7),

$$\xi_i(g)=P_0(g)*P_1(g_1)*\cdots*P_i(g_i),$$

and (3.1) holds. The smoothness of the mapping $G\times G\to G:(g,h)\to g*h$ follows from (3.2) and C1. Hence, $(G,*)$ is a Lie group. The mapping φ is a Lie group homomorphism by (3.3),(3.4), and moreover, $\varphi(g)=(e,e)$ implies $g=\rho_+(g)\rho_-(g)=e$, i.e., φ is an injective homomorphism. ∎

EXAMPLE 1. Let $\tilde{G}_\pm\subset G_\pm$ be Lie subgroups such that

$$\tilde{G}_+\tilde{G}_-=G,\qquad \tilde{G}_+\cap\tilde{G}_-=e. \tag{3.5}$$

Set $g=\rho_+(g)\rho_-(g)$, where $\rho_+(g)\in\tilde{G}_+$, $\rho_-(g)\in\tilde{G}_-$ are uniquely defined. Clearly (3.3) holds in this situation. Let us denote by $L(G)$ a Lie algebra of G (i.e., the tangent space to G at e endowed with a Lie bracket operation; see [19]). Then (3.5) implies

$$L(G)=L(\tilde{G}_+)\oplus L(\tilde{G}_-). \tag{3.6}$$

This situation [which we call the Adler-Kostant-Symes (AKS) case] arises in the most practically important situations (see e.g. [6, 10, 14, 20, 21]).

In general (unlike the AKS case) it is rather difficult to verify (3.3),(3.4) directly. We proceed as follows. To verify (3.3),(3.4) it suffices to establish that the corresponding tangent maps $(T_e(\rho_+),T_e(\rho_-^{-1})):L((G,*))\to(L(G_+)\times L(G_-))$ define a Lie-algebra homomorphism (see e.g. [19]). First of all we describe a Lie bracket operation on $L(G)$ corresponding to the Lie group structure $(G,*)$. Let us denote by $[\ ,\ ],[\ ,\ ]'$ standard Lie bracket operations on $L(G)$ and $L(G_+)\times L(G_-)$, respectively (i.e., $[(\xi_+,\xi_-),(\eta_+,\eta_-)]'=[\xi_+,\eta_+]+[\xi_-,\eta_-]$, $\xi_+,\eta_+\in L(G_+)$, $\xi_-,\eta_-\in L(G_-)$), and by $[\ ,\ ]''$ the Lie bracket operation on $L(G)$ corresponding to $(G,*)$. Let, further, $\varphi: G\to G_+\times G_-$, $\varphi(g)=(\rho_+(g),\rho_-(g)^{-1})$, $\Psi:G_+\times G_-\to G$, $\Psi(g_+,g_-)=g_+g_-^{-1}$. Clearly $\Psi\circ\varphi(g)=g$, $g\in G$. Hence $T_e(\Psi)T_e(\varphi)\xi=\xi$, $\xi\in L(G)$. Set

$$\xi_\pm=T_e(\rho_\pm)\cdot\xi. \tag{3.7}$$

We have $T_e(\varphi)\cdot\xi=(\xi_+,-\xi_-)$, and given $(\eta_1,\eta_2)\in L(G_+)\times L(G_-)$, we have $T_e(\Psi)\,(\eta_1,\eta_2)=\eta_1-\eta_2$. Under the assumptions of Proposition 4, $T_e(\varphi)$ is a Lie-algebra homomorphism. Consequently, given $\xi,\eta\in L(G)$, we have $[\xi,\eta]''=T_e(\Psi)\circ T_e(\varphi)\cdot[\xi,\eta]''=T_e(\Psi)\cdot[T_e(\varphi)\cdot\xi,T_e(\varphi)\cdot\eta]'=T_e(\Psi)\cdot([(\xi_+,-\xi_-),(\eta_+-\eta_-)]')=T_e(\Psi)\cdot([\xi_+,\eta_+],[\xi_-,\eta_-])=[\xi_+,\eta_+]-[\xi_-,\eta_-]$, where we have used the designations (3.7). Let us introduce a linear operator $R:L(G)\to L(G)$ by

$$R\xi=\xi_+-\xi_-, \tag{3.8}$$

[see (3.7)]. We have

$$\begin{aligned}[\xi,\eta]''&=[\xi_+,\eta_+]-[\xi_-,\eta_-]\\&=\tfrac{1}{2}([\xi_++\xi_-,\eta_+-\eta_-]+[\xi_+-\xi_-,\eta_++\eta_-])\\&=\tfrac{1}{2}([\xi,R\eta]+[R\xi,\eta]).\end{aligned}$$

Here we have used $T_e(\Psi)\circ T_e(\varphi)\cdot\xi=\xi=\xi_++\xi_-$. Thus, we have proved

Proposition 5. *Under the assumptions of Proposition 4, let $R:L(G)\to L(G)$ be a linear operator as defined in (3.7),(3.8). Then the Lie bracket $[\ ,\ ]_R$ on $L(G)$ corresponding to the Lie group $(G,*)$ in terms of the standard Lie bracket $[\ ,\ ]$ takes the form*

$$[\xi,\eta]_R=\tfrac{1}{2}([R\xi,\eta]+[\xi,R\eta]),\qquad \xi,\eta\in L(G). \tag{3.9}$$

Let us write down a condition that the mapping $T_e(\rho_+):(L(G),[\ ,\ ]_R)\to L(G_+):\xi\to\xi_+$ is a Lie-algebra homomorphism. We have [see (3.7)]

$$T_e(\rho_+)\cdot[\xi,\eta]_R=[\xi_+,\eta_+],\qquad \xi,\eta\in L(G).$$

But by (3.7),(3.8), $T_e(\rho_+)=\frac{1}{2}(R+E)$, where $E:L(G)\to L(G)$ is an identity operator. Hence

$$(R+E)(2[\xi,\eta]_R)=[(R+E)\xi,(R+E)\eta],$$

or

$$R([R\xi,\eta]+[\xi,R\eta])=[R\xi,R\eta]+[\xi,\eta],\qquad \xi,\eta\in L(G). \quad (3.10)$$

Thus, we have proved

Proposition 6. *Let $\rho_{\pm}$ be a factorization on G satisfying condition* C1, *and $R: L(G)\to L(G)$ be a linear operator defined by* (3.7), (3.8). *For condition* C2 *to hold it is necessary that the identity* (3.10) *hold.*

We call (3.10) (following [15]) the modified Yang-Baxter equation. It is a remarkable fact (which is essentially due to Semenov-Tian-Shansky [15]) that the converse assertion is also true. Firstly we give a description of solutions to (3.10).

Let $R: L(G)\to L(G)$ be a linear operator. Consider vector subspaces in $L(G)$

$$C_{\pm}=\operatorname{Im}(R\pm E),\qquad D_{\pm}=\operatorname{Ker}(R\mp E). \quad (3.11)$$

Clearly $D_{\pm}\subset C_{\pm}$. Let us define a linear mapping $\vartheta: C_+/D_+\to C_-/D_-$,

$$\vartheta(\pi_+(R+E)x)\triangleq\pi_-(R-E)x,\qquad x\in L(G), \quad (3.12)$$

where $\pi_{\pm}: C_{\pm}\to C_{\pm}/D_{\pm}$ are the canonical projections onto factor spaces $C_{\pm}/D_{\pm}$. We call ϑ (following Belavin and Drinfeld [2]) the generalized Cayley transform of R. The correctness of the definition (3.12) is almost obvious from the following commutative diagram:

$$\begin{tikzcd} C_+/D_+ \arrow[rr, "\vartheta"] & & C_-/D_- \\ C_+ \arrow[u, "\pi_+"] & & C_- \arrow[u, "\pi_-"'] \\ & L(G) \arrow[ul, "(R+E)/2"] \arrow[ur, "(R-E)/2"'] & \end{tikzcd} \quad (3.13)$$

Theorem 1. *Let R be a linear operator which maps $L(G)$ into $L(G)$. Then R is a solution to the Yang-Baxter equation* (3.10) *if and only if the* *following conditions hold*:

(i) *the vector spaces* $C_{\pm}$ *(see* (3.11)*) are Lie subalgebras in* $L(G)$ *endowed with the standard Lie algebra structure*;

(ii) *the vector spaces* $D_{\pm}$ *(see* (3.11)*) are ideals in Lie algebras* $C_{\pm}$, *respectively*;

(iii) *the generalized Cayley transform* ϑ *of* R *(see* (3.12),(3.13)*) is a Lie-algebra homomorphism.*

PROPOSITION 7. *Let* R *be a solution to* (3.10). *Then* $[\ ,\ ]_R$ *(see* (3.9)*) defines a second Lie-algebra structure on* $L(G)$, *and the mappings* $\frac{1}{2}(R \pm E):(L(G),[\ ,\ ]_R) \to (L(G),[\ ,\ ])$ *are Lie-algebra homomorphisms.*

Proof. We have by (3.10)

$$\frac{R \pm E}{2}([\xi,\eta]_R) = \left[\left(\frac{R \pm E}{2}\xi, \frac{R \pm E}{2}\eta\right]\right] \qquad \forall \xi,\eta \in L(G). \quad (3.14)$$

To prove that $[\ ,\ ]_R$ defines a Lie-algebra structure on $L(G)$ we must verify the Jacobi identity

$$\Delta = [\xi,[\eta,\zeta]_R]_R + [\eta,[\zeta,\xi]_R]_R + [\zeta,[\xi,\eta]_R]_R = 0, \qquad (3.15)$$

$\xi,\eta,\zeta \in L(G)$. By (3.14) and the Jacobi identity for the standard bracket $[\ ,\ ]$ we have $\Delta \subset \mathrm{Ker}(R+E) \cap \mathrm{Ker}(R-E) = \{0\}$. This proves (3.15). The second claim of Proposition 7 follows now from (3.14). ■

Proof of Theorem 1. Let R be a solution to (3.10). By Proposition 7,

$$C_{\pm} = \mathrm{Im}\left(\frac{R \pm E}{2}\right)$$

are Lie subalgebras in $(L(G),[\ ,\ ])$, and

$$D_{\pm} = \mathrm{Ker}\left(\frac{R \mp E}{2}\right)$$

are ideals in $(L(G),[\ ,\ ]_R)$. But clearly

$$D_{\pm}=\frac{R\pm E}{2}D_{\pm}.$$

Thus, $D_{\pm}$ are also ideals in

$$\frac{R\pm E}{2}L(G)=C_{\pm}.$$

Now ϑ is a Lie-algebra homomorphism by (3.15) and Proposition 7.

Conversely, let $R: L(G)\to L(G)$ be a linear operator such that conditions (i), (ii), (iii) hold. Let, further, $\xi,\eta\in L(G)$. Denote by $[\ ,\ ]_{\pm}$ the Lie brackets in the factor algebras $C_{\pm}/D_{\pm}$ induced from $[\ ,\ ]$. By (i), (ii), (iii) we have

$$\pi_{\pm}([x_{\pm},y_{\pm}])=[\pi_{\pm}x_{\pm},\pi_{\pm}y_{\pm}]_{\pm},\qquad x_{\pm},y_{\pm}\in C_{\pm},\tag{3.16}$$

$$\vartheta([a,b]_{+})=[\vartheta(a),\vartheta(b)]_{-},\qquad a,b\in C_{+}/D_{+},\tag{3.17}$$

where $\pi_{\pm}: C_{\pm}\to C_{\pm}/D_{\pm}$ are canonical projections. Let, further, $\xi,\eta\in L(G)$. Since C_{+} is a Lie subalgebra of $L(G)$, we have

$$[(R+E)\xi,(R+E)\eta]=(R+E)\zeta\tag{3.18}$$

for some $\zeta\in L(G)$. By (3.13), (3.16), (3.17), (3.18), we obtain $\vartheta\circ\pi_{+}([(R+E)\xi,(R+E)\eta])=\vartheta\circ\pi_{+}\circ(R+E)\zeta=\pi_{-}\circ(R-E)\zeta=$ $\vartheta\circ\pi_{+}\circ[(R+E)\xi,\vartheta\circ\pi_{+}\circ(R+E)\eta]_{-}=[\pi_{-}\circ(R-E)\xi,\pi_{-}\circ(R-E)\eta]_{-}=$ $\pi_{-}([(R-E)\xi,(R-E)\eta])$, i.e.,

$$(R-E)\zeta-[(R-E)\xi,(R-E)\eta]\in\operatorname{Ker}(R+E)=D_{-}.\tag{3.19}$$

Now (3.18), (3.19) imply

$$[R\xi,\eta]+[\xi,R\eta]-\zeta\in\operatorname{Ker}(R+E).\tag{3.20}$$

Applying $R+E$ to (3.20) and taking into account (3.18), we obtain

$$(R+E)([R\xi,\eta]+[\xi,R\eta])=[(R+E)\xi,(R+E)\eta].$$

But this equation is exactly (3.10). ■

REMARK. Theorem 1 is essentially due to Belavin and Drinfeld [2]. But a detailed proof has not been published.

EXAMPLE 2. Let $L(G)=L_{+}\oplus L_{-}$ (direct sum of vector subspaces), where $L_{\pm}$ are Lie subalgebras of $L(G)$. Let, further, $\pi_{\pm}: L(G)\to L_{\pm}$ be a projection of $L(G)$ onto $L_{\pm}$ along $L_{\mp}$. Consider a linear operator R: $L(G)\to L(G)$,

$$R=\pi_{+}-\pi_{-}.$$

Clearly [see (3.11)] $D_{\pm}=L_{\pm}=C_{\pm}$. Thus, in this case $C_{+}/D_{+}=C_{-}/D_{-}=\{0\}$. Hence the Cayley transform ϑ of R [see (3.12)] is a Lie-algebra homomorphism. By Theorem 1 R is a solution to (3.10). This solution corresponds to AKS-case factorization, which we considered in Example 1 [see (3.8)].

EXAMPLE 3. Consider a slightly more general situation. Let $L(G)=L_{+}\oplus L_{0}\oplus L_{-}$ (direct sum of vector subspaces), and L_0 be a commutative subalgebra of $L(G)$. Let, further, $[L_0, L_{\pm}]\subset L_{\pm}$. Denote by $\pi_{\pm}$ the projection of $L(G)$ onto $L_{\pm}$ along $L_0\oplus L_{\mp}$. Set

$$R=\pi_{+}-\pi_{-}. \tag{3.21}$$

In this case [see (3.11)] $C_{\pm}=L_0\oplus L_{\pm}$, $D_{\pm}=L_{\pm}$. The conditions $[L_0, L_{\pm}]\subset L_{\pm}$ imply that $D_{\pm}$ is an ideal in $C_{\pm}$. Moreover, $C_{\pm}/D_{\pm}\approx L_0$. But L_0 is a commutative Lie-algebra and consequently ϑ is a Lie-algebra homomorphism. Thus, (3.21) is a solution to (3.10).

Given a solution R to (3.10), let ϑ be its Cayley transform. We use the symbols $C_{\pm}, D_{\pm}, \pi_{\pm}, [\ ,\ ]_{\pm}$ as they are described in (3.11), (3.12), (3.16), (3.17). We denote by $\Psi_{\pm}: L_R=(L(G),[\ ,\ ]_R)\to C_{\pm}$ the Lie-algebra homomorphisms.

THEOREM 2. *Let R be a solution to the Yang-Baxter equation* (3.10). *There exists an open neighborhood U of the identity element of G and smooth functions $\rho_{\pm}: U\to G(C_{\pm})$ such that conditions* C1,C2 *hold. Here $G(C_{\pm})$ is a connected Lie subgroup of the Lie group G with the Lie algebra $C_{\pm}$ (see e.g.* [19]).

Proof. By Proposition 7 a linear operator R determines a second Lie-algebra structure $[\ ,\ ]_R$ on $L(G)$. Denote by G_R the corresponding simply connected Lie group, and by $\gamma_{\pm} : G_R \to G(C_{\pm})$ the Lie-group homomorphism such that $T_{e_R}(\gamma_{\pm}) = \Psi_{\pm}$ [19], where e_R is the identity element in G_R. Consider a mapping $\Psi : G_R \to G$, $\Psi(h) = \gamma_+(h)\gamma_-(h)^{-1}$. We have for $x \in L(G) = T_{e_R}(G_R) = T_e(G)$

$$T_e(\Psi)\cdot x = \Psi_+(x) - \Psi_-(x) = x.$$

Hence by the implicit-function theorem (see e.g. [19]) there are neighborhoods $\tilde{U}, U$ of the identity elements e_R, e, respectively, such that $\Psi|\tilde{U}$ is a diffeomorphism of $\tilde{U}$ onto U. Given $g \in U$, set $\rho_+(g) = \gamma_+(\Psi^{-1}(g))$, $\rho_-(g) = [\gamma_-(\Psi^{-1}(g))]^{-1}$. We have $\rho_+(g)\rho_-(g) = \gamma_+(\Psi^{-1}(g))\cdot[\gamma_-(\Psi^{-1}(g))]^{-1} = \Psi \circ \Psi^{-1}(g) = g$. Clearly $\rho_{\pm}(e) = e$. Let, further, $g_1, g_2 \in U$ be such that $\varkappa(g_1, g_2) = \rho_+(g_1)g_2\rho_-(g_1) \in U$. To verify C2 we must show $\rho_+(\varkappa(g_1, g_2)) = \rho_+(g_1)\rho_+(g_2)$. Let us denote by $*$ the group operation on G_R. We have $\varkappa(g_1, g_2) = \gamma_+(\Psi^{-1}(g_1))\gamma_+(\Psi^{-1}(g_2))[\gamma_-(\Psi^{-1}(g_2))]^{-1}[\gamma_-(\Psi^{-1}(g_1))]^{-1} = \gamma_+(\Psi^{-1}(g_1) * \Psi^{-1}(g_2))[\gamma_-(\Psi^{-1}(g_1)) * \Psi^{-1}(g_2))]^{-1} = \Psi(\Psi^{-1}(g_1) * \Psi^{-1}(g_2))$ by definition of Ψ and using the fact that $\gamma_{\pm} : G_R \to G(C_{\pm})$ are Lie-group homomorphisms. Now $\rho_+(\varkappa(g_1, g_2)) = \gamma_+(\Psi^{-1}(\varkappa(g_1, g_2))) = \gamma_+(\Psi^{-1}(g_1) * \Psi^{-1}(g_2)) = \gamma_+(\Psi^{-1}(g_1))\cdot\gamma_+(\Psi^{-1}(g_2)) = \rho_+(g_1)\rho_+(g_2)$. ∎

Let $G(D_+)$ be a connected Lie subgroup of the Lie group $G(C_{\pm})$ with the Lie algebra $D_{\pm}$. Then $G(D_{\pm})$ is a normal subgroup of $G(C_{\pm})$ (see e.g. [19]). We suppose that $G(D_{\pm})$ is a closed subgroup of $G(C_{\pm})$ (this is true e.g. when $G(C_{\pm})$ is a simply connected Lie group [3]). Then a factor group $B_{\pm} = G(C_{\pm})/G(D_{\pm})$ and a Lie-group homomorphism $\beta_{\pm} : G(C_{\pm}) \to B_{\pm}$ are defined. Moreover, $T_e(\beta_{\pm}) = \pi_{\pm}$. If B_+ is s simply connected Lie group (this is true e.g. when $G(C_{\pm})$ is a simply connected Lie group [3]), then we can define a Lie-group homomorphism $\alpha : B_+ \to B_-$ such that $T_e(\alpha) = \vartheta$. Clearly we have $T_e(\alpha \circ \beta_+ \circ \gamma_+) = \vartheta \circ \pi_+ \circ \Psi_+$ and $T_e(\beta_- \circ \gamma_-) = \pi_- \circ \Psi_-$. By (3.13) we obtain

$$\alpha \circ \beta_+ \circ \gamma_+ = \beta_- \circ \gamma_-. \tag{3.22}$$

THEOREM 3. *Let $\rho_{\pm}$ be a factorization described in Theorem 2 and defined on a connected open neighborhood U of e. If $G(D_{\pm})$ is a closed subgroup of $G(C_{\pm})$ and the Lie group $G(C_+)/G(D_+)$ is simply connected, then for the factorization $\rho_{\pm}$ we have*

$$\alpha \circ \beta_+ \circ \rho_+(g) = [\beta_- \circ \rho_-(g)]^{-1}. \tag{3.23}$$

The smooth functions $\rho_{\pm}: U \to G(C_{\pm})$ *satisfying* C1 *and* (3.23) *are determined uniquely.*

PROPOSITION 8. *Given* $x \in L(G)$, *there are unique* $x_{\pm} \in C_{\pm}$ *such that*

$$x = x_+ - x_-, \qquad \vartheta \circ \pi_+(x_+) = \pi_-(x_-). \tag{3.24}$$

Proof. Set $x_{\pm} = \Psi_{\pm}(x)$. We have $x_+ - x_- = (R+E)/2x - (R-E)/2x = x$. Further, $\vartheta \circ \pi_+(x_+) = \vartheta \circ \pi_+ \circ \Psi_+(x) = \pi_- \circ \Psi_-(x) = \pi_-(x_-)$ [see (3.13)]. Let $x = y_+ - y_-$, where $y_{\pm}$ satisfy (3.24). Because $y_{\pm} \in C_{\pm} = \operatorname{Im}\Psi_{\pm}$, we have $y_{\pm} = \Psi(z_{\pm})$, $z_{\pm} \in L(G)$. By (3.24) $\vartheta \circ \pi_+ \circ \Psi_+(z_+) = \pi_- \circ \Psi_-(z_-)$. Hence, using (3.13), we obtain $\vartheta \circ \pi_+ \circ \Psi_+(z_+ - z_-) = \pi_- \circ \Psi_-(z_+ - z_-) = 0$. In other words, $\Psi_{\pm}(z_+ - z_-) \in D_{\pm} = \operatorname{Ker}\Psi_{\mp}$, i.e.,

$$\Psi_+(z_+) = \Psi_+(z_-) + t_-, \qquad \Psi_-(t_-) = 0,$$

$$\Psi_-(z_-) = \Psi_-(z_+) + t_+, \qquad \Psi_+(t_+) = 0.$$

Further, $x = \Psi_+(z_+) - \psi_-(z_-) = (\psi_+ - \psi_-)(z_+) - t_+ = z_+ - t_+$. Thus, applying ψ_+, we obtain $\psi_+(x) = \psi_+(z_+)$. Similarly $\psi_-(x) = \psi_-(z_-)$, i.e. $x_{\pm} = y_{\pm}$. ∎

Proof of Theorem 3. Making use of the notation of the proof of Theorem 2, we have

$$\rho_+(g) = \gamma_+(\psi^{-1}(g)), \qquad \rho_-(g) = [\gamma_-(\psi^{-1}(g))]^{-1}$$

Hence

$$\alpha \circ \beta_+ \circ \rho_+(g) = \alpha \circ \beta_+ \circ \gamma_+(\psi^{-1}(g)) = \beta_- \circ \gamma_-(\psi^{-1}(g))$$

by (3.22). Thus, (3.23) holds. Let $\tau_{\pm}: U \to G(C_{\pm})$ be another smooth factorization satisfying (3.23) and C1. Given $g \in U$, we have $\rho_+(g)\rho_-(g) = \tau_+(g)\tau_-(g)$ or $\tau_+(g)^{-1}\rho_+(g) = \tau_-(g)\rho_-(g)^{-1}$. Consider a smooth function $\mu: U \to G(C_+) \cap G(C_-)$, $\mu(g) = \tau_+(g)^{-1}\rho_+(g)$. We have by (3.23) $\alpha \circ \beta_+ \circ \mu(g) = \beta_- \circ \mu(g)$. Consider a set $H \subset G$, $H \triangleq \{g \in G(C_+) \cap G(C_-): \alpha \circ \beta_+(g) = \beta_-(g)\}$. It is an elementary exercise in Lie-group theory (see also [3]) to verify that H is a Lie subgroup of G with a Lie algebra $L(H) = \{x \in C_+ \cap C_+ : \vartheta \circ \pi_+(x) = \pi_-(x)\}$. Given $x \in L(H)$, we have $0 = x - x$. By Proposition 8 it follows that $x = 0$, i.e. $L(H) = 0$. This implies that H is a discrete subgroup of $G_+ \cap G_-$. A smooth function μ is defined on the connected set U and takes its values in a discrete set H. Thus, $\mu(g) = \mu(e) = e$, that is, $\tau_+(g) = \rho_+(g)$. ■

REMARK. Theorem 3 is essentially due to Semenov-Tian-Shansky [15], though a proof has not been published.

4. QR-TYPE ALGORITHMS AND DYNAMICAL SYSTEMS

Since the pioneering paper by Symes [18] (see also [10]) there has been considerable interest in the description of relationship between QR-type algorithms and dynamical systems. One of the main ideas behind this approach is to try to use known methods of a numerical integration of dynamical systems for the solution of eigenvalue problems. Several authors [6, 7, 13, 20, 21] considered rather general situations of this type (within the AKS case). Here we consider a group-theoretic scheme for the factorizations associated with the Yang-Baxter equation.

Lie L^* be a vector space of linear functionals on a Lie algebra $L(G)$ of a Lie group G. We denote by $\langle\ ,\ \rangle$ the usual pairing between L^* and $L(G)$, i.e., given $\alpha \in L^*$, $\xi \in L(G)$, we have

$$\langle \alpha, \xi \rangle \triangleq \alpha(\xi). \tag{4.1}$$

We assume the notion of adjoint representation of G on $L(G)$ (see e.g. [19]). For the case of a matrix group G we have

$$\operatorname{Ad}(g) \cdot \xi = g \xi g^{-1}, \qquad g \in G, \quad \xi \in L(G). \tag{4.2}$$

Given $\xi \in L(G)$, denote by $\operatorname{ad} \xi$ the linear operator on $L(G)$

$$\operatorname{ad} \xi(\eta) = [\xi, \eta], \qquad \eta \in L(G). \tag{4.3}$$

Making use of (4.1)–(4.3), we can define a coadjoint representation of G on L^*:

$$\begin{gathered}
\langle \operatorname{Ad}^*(g) \cdot \alpha, \xi \rangle \triangleq \langle \alpha, \operatorname{Ad}(g^{-1}) \cdot \xi \rangle, \\
\langle \operatorname{ad}^*(\eta) \cdot \alpha, \xi \rangle \triangleq -\langle \alpha, \operatorname{ad} \eta(\xi) \rangle, \\
g \in G, \qquad \alpha \in L^*, \qquad \xi, \eta \in L(G).
\end{gathered} \tag{4.4}$$

We call a smooth function φ on L^* a Casimir function if

$$\mathrm{ad}^*(\nabla\varphi(\alpha))\cdot\alpha = 0 \qquad \forall\alpha\in L^*, \tag{4.5}$$

where $\nabla\varphi(\alpha)\in L(G)$ is by definition

$$\langle\beta,\nabla\varphi(\alpha)\rangle = \lim_{t\to 0}\frac{\varphi(\alpha+t\beta)-\varphi(\alpha)}{t}, \qquad \alpha,\beta\in L^*, \quad t\in\mathbb{R}.$$

Given a smooth function φ on $\mathbb{R}\times L^*$, we denote by φ_t the smooth function on L^* defined by

$$\varphi_t(\alpha)\triangleq\varphi(t,\alpha), \qquad \alpha\in L^*.$$

Let R be a solution to the Yang-Baxter equation (3.10), and φ be a smooth function on $\mathbb{R}\times L^*$ such that φ_t is a Casimir function on L^* for every $t\in\mathbb{R}$. Consider a nonstationary dynamical system on L^*:

$$\dot\alpha(t) = -\,\mathrm{ad}^*\left(\frac{R}{2}\nabla\varphi_t(\alpha(t))\circ\alpha(t)\right), \tag{4.6}$$

$\alpha(0)=\alpha_0$.

THEOREM 4. *Let $\rho_\pm : U\to G(C_\pm)$ be the factorization associated with R (see Theorems 2,3), where U is an open connected neighborhood of the identity element of G. Let, further,*

$$\delta(t)\triangleq\exp\left(\int_0^t\nabla\varphi_\tau(\alpha_0)\,d\tau\right)=\rho_+(t)\rho_-(t) \tag{4.7}$$

be a factorization satisfying C1, C2, (3.23) *and defined at least on $[0,t^*[$, where $t^*>0$ is such that $\delta(t)\in U$, $0\leqslant t<t^*$. Then the solution to* (4.6) *is defined at least on $[0,t^*[$ and takes the form*

$$\alpha(t)=\mathrm{Ad}^*\left(\rho_+(t)^{-1}\right)\alpha_0=\mathrm{Ad}^*(\rho_-(t))\cdot\alpha_0. \tag{4.8}$$

The proof of Theorem 4 has an analytic flavor, and we don't give it here. Let $\langle\langle\ ,\ \rangle\rangle$ be a nondegenerate bilinear Ad-invariant form on $L(G)$. It enables us to identify L^* with $L(G)$ via

$$\langle \alpha, x \rangle = \langle\langle \alpha, x \rangle\rangle, \qquad \alpha \in L^*, \qquad x \in L(G).$$

Under this identification we have $\mathrm{ad}^* \approx \mathrm{ad}$, and (4.4) takes the form

$$\dot{\alpha}(t) = \left[\alpha(t), \frac{R}{2}\nabla\varphi_t(\alpha(t))\right], \qquad \alpha(0) = \alpha_0 \in L(G). \tag{4.6a}$$

For the AKS case (see Example 2) we arrive at the standard situation [6, 14, 20].

In connection with (4.5) we mention the following fact.

PROPOSITION 9. *Let φ_i, $i = 1,2$, be Casimir functions on L^*. Then*

$$[\nabla\varphi_i(\alpha), \nabla\varphi_2(\alpha)] = 0, \qquad \alpha \in L^*. \tag{4.9}$$

If G is a connected Lie group, then

$$\nabla\varphi_1(\mathrm{Ad}^*(g)\cdot\alpha) = \mathrm{Ad}(g)\cdot\nabla\varphi_1(\alpha), \qquad g \in G, \quad \alpha \in L^*. \tag{4.10}$$

Proof. Firstly we prove (4.10). It suffices to consider the case $g = \exp(\xi)$, $\xi \in L(G)$ (see e.g. [19]). Let $\Psi(t) = \mathrm{Ad}^*(\exp(\xi t))\cdot\alpha$, $\varepsilon(t) = \varphi(\Psi(t))$, $t \in \mathbb{R}$. We have $\dot{\Psi}(t) = \mathrm{ad}^*(\xi)\Psi(t)$, whence $\dot{\varepsilon}(t) = \langle \mathrm{ad}^*(\xi)\Psi(t), \nabla\varphi(\Psi(t))\rangle = -\langle \mathrm{ad}^*(\nabla\varphi(\Psi(t)))\cdot\Psi(t), \xi\rangle = 0$. Here we have used (4.4),(4.5). Thus, $\varphi(\Psi(1)) = \varphi(\Psi(0))$, that is, $\varphi(\mathrm{Ad}^*(g)\cdot\alpha) = \varphi(\alpha)$, $g \in G$, $\alpha \in L^*$. Given $t \in \mathbb{R}$, $g \in G$, $\alpha, \beta \in L^*$, we have

$$\varphi(\mathrm{Ad}^*(g)\cdot(\alpha + t\beta)) = \varphi(\alpha + t\beta). \tag{4.11}$$

Differentiating (4.11) with respect to t, we obtain

$$\varphi\langle \mathrm{Ad}^*(g)\cdot\beta, \nabla\varphi(\mathrm{Ad}^*(g)\cdot\alpha)\rangle = \langle \beta, \nabla\varphi(\alpha)\rangle,$$

i.e.

$$\mathrm{Ad}(g^{-1})\nabla\varphi(\mathrm{Ad}^*(g)\cdot\alpha) = \nabla\varphi(\alpha).$$

Now we prove (4.9). Let $\Psi(t) = \mathrm{Ad}(\exp(t\nabla\varphi_1(\alpha)))$. We have $\Psi(t)\nabla\varphi_2(\alpha) =$

$\nabla\varphi_2(\mathrm{Ad}^*(\exp(t\nabla\varphi_1(\alpha)))\cdot\alpha)=\varepsilon$ by (4.10). Further,

$$\varepsilon=\nabla\varphi_2\big(\exp\big(\mathrm{ad}^*\big(t\nabla\varphi_1(\alpha)\big)\cdot\alpha\big)=\nabla\varphi_2(\alpha)$$

since $\mathrm{ad}^*(\nabla\varphi_1(\alpha))\cdot\alpha=0$. Hence $\Psi(t)\cdot\nabla_{\varphi 2}(\alpha)=\nabla\varphi_2(\alpha)$. Differentiating this equality with respect to t, we obtain (4.9). ∎

Let $g_0\in G$, and $\Gamma: C(g_0)\to L^*$ be a function such that $\Gamma(g_0)=\alpha_0$, $\Gamma(gg_0g^{-1})=\mathrm{Ad}^*(g)\circ\alpha_0$, $g\in G$. Consider the functions P_i, $i=0,1\dots$; $P_i: C(g_0)\to G$,

$$P_i(g)=\exp(\nabla f_i(\Gamma(g))), \tag{4.12}$$

where f_i are Casimir functions on L^*.

Proposition 10. *If G is a connected Lie group, then*

$$P_i\big(gg_0g^{-1}\big)=g\cdot P_i(g_0)\cdot g^{-1},\qquad g\in G,\quad i=0,1,\dots. \tag{4.13}$$

Proof. By (4.10),

$$\begin{aligned}P_i\big(gg_0g^{-1}\big)&=\exp\big(\nabla f_i(\mathrm{Ad}^*(g)\cdot\alpha_0)\big)\big)\\&=\exp\big(\mathrm{Ad}(g)\nabla f_i(\alpha_0)\big)=g\exp\big(\nabla f_i(\alpha_0)\big)\cdot g^{-1}=g\cdot P_i(g_0)\cdot g^{-1}.\end{aligned}$$

Having a sequence P_i satisfying (4.13) and the factorization associated with a solution R to (3.10), we can construct the iterations $g_0, g_1,\dots,$ of a QR-type algorithm [see (2.7), (2.8)].

Theorem 5. *Let $\varphi:\mathbb{R}\times L^*\to\mathbb{R}$ be a smooth function such that:*

(i) φ_t *is a Casimir function for every* $t\in\mathbb{R}$;
(ii) $\int_{i-1}^{i}\nabla\varphi_\tau(\alpha_0)\,d\tau=\nabla f_i(\alpha_0)$, $i=1,2,\dots$; $\nabla f_0(\alpha_0)=0$.

Let the solution $\alpha(t)$ to (4.6) be defined at least on $[0,k]$ for some positive integer k. Then the iterates $g_0, g_1, g_i,\dots$ of a QR-type algorithm are defined at least for $i=0,1,\dots,k$, and

$$\Gamma(g_i)=\alpha(i), \tag{4.14}$$

$$Q_i=\rho_+(i),\quad R_i=\rho_-(i),\qquad i=0,1,\dots,k, \tag{4.15}$$

where Q_i, R_i are defined as in (2.10) and $\rho_\pm(t)$ are defined as in (4.7).

Proof. By Proposition 9 we have for $0 \leqslant i \leqslant k$

$$\exp\left(\int_0^i \nabla \varphi_\tau(\alpha_0)\, d\tau\right) = \exp\left(\int_{i-1}^i \nabla \varphi_\tau(\alpha_0)\, d\tau\right)\exp\left(\int_{i-2}^{i-1} \nabla \varphi_\tau(\alpha_0)\, d\tau\right)$$
$$\times \cdots \times \exp\left(\int_0^1 \nabla \varphi_\tau(\alpha_0)\, d\tau\right).$$

Hence by (ii), (4.7),(4.13) we obtain

$$\rho_+(i)\rho_-(i) = P_i(g_0)P_{i-1}(g_0)\cdots P_1(g_0) = \xi_i(g_0)$$

because $\nabla f_0(\alpha_0) = 0$, i.e. $P_0(g_0) = e$. On the other hand, by Proposition 3.4 and Theorem 2 we have $\xi_i(g) = Q_i R_i$ and $\rho_+(\xi_i(g_0)) = Q_i$. Thus, (4.15) holds. Further, by Proposition 3 and Theorem 4, $g_i = Q_i^{-1} g_0 Q_i$, $\alpha(i) = \mathrm{Ad}^*(Q_i^{-1})\cdot\alpha_0$. Further, $\Gamma(g_i) = \Gamma(Q_i^{-1} g_0 Q_i) = \mathrm{Ad}^*(Q_i^{-1})\cdot\alpha_0$, i.e., (4.14) holds. ∎

5. EXAMPLES

EXAMPLE 4 (Continuation of Example 3). Let $G = \mathrm{GL}(r, \mathbb{R})$, and $L(G)$ be the Lie algebra of n-by-n real matrices. Consider in $L(G)$ the Lie subalgebras $C_\pm$, where C_+ (C_-) is the Lie algebra of lower (upper) triangular real n-by-n matrices. Let, further, $D_\pm$ be the ideal in $C_\pm$, where D_+ (D_-) is the ideal in C_+ (in C_-) of strictly lower (upper) triangular matrices.

We denote by C_0 the set of diagonal matrices in $L(G)$, and by $\pi_\pm : L(G) \to C_\pm$, $\pi_0 : L(G) \to C_0$ the corresponding projections. By Theorem 1

$$R = \pi_+ - \pi_- \tag{5.1}$$

is a solution to (3.10). In our situation $G(C_+)$ $[G(C_-)]$ is the Lie group of lower [upper] triangular matrices with positive entries on the main diagonal, and $G(D_+)$ $[G(D_-)]$ is the Lie group of unit lower [upper] triangular matrices. Clearly we can identify $G(C_\pm)/G(D_\pm)$ with the Lie group $G(C_0)$ of diagonal matrices with positive entries. Making use of the notation of Theorem 3, we have, as is easily seen,

$$\vartheta(z) = -z, \quad z \in C_0, \qquad \alpha(x) = x^{-1}, \quad x \in G(C_0),$$
$$\beta_\pm(y_\pm) = \pi_0(y_\pm), \qquad y_\pm \in G(C_\pm).$$

Thus, by Theorem 3 we have the following conditions for the factorization $\rho_{\pm}: U \to G(C_{\pm})$ corresponding to the solution (5.1) of (3.10):

$$g = \rho_+(g)\rho_-(g), \qquad \pi_0(\rho_+(g)) = \pi_0(\rho_-(g)); \tag{5.2}$$

see (3.23). Here U is a connected neighborhood of the identity element in G.

PROPOSITION 11. *Given $g \in G$, the factorization* (5.2) *exists if and only if g has only positive leading principal minors.*

Proof. Proceed by induction on n. For the case $n = 1$ the assertion is clearly true. Let g be an $(n+1)$-by-$(n+1)$ nonsingular matrix of the form

$$g = \begin{bmatrix} A & x \\ y & a \end{bmatrix},$$

where A is an n-by-n matrix, x is an n-by-1 column, y is 1 by n row and a is 1 by 1 entry. Let, further,

$$g = \begin{bmatrix} A_- & 0 \\ y_- & a_- \end{bmatrix} \begin{bmatrix} A_+ & x_+ \\ 0 & a_+ \end{bmatrix}$$

be a factorization (5.2) with the same block structure. Clearly we have

$$A = A_- A_+, \qquad x = A_- x_+, \qquad y = y_- A_+, \qquad a = y_- x_+ + a_- a_+. \tag{5.3}$$

By the inductive assumption we see that A has only positive leading principal minors, and moreover, by (5.3) and (5.2), $a_+ = a_- = b$ and

$$a - yA_+^{-1}A_-^{-1}x = a - yA^{-1}x = b^2 > 0.$$

But by Frobenius's formula we have $\det g = \det A(a - yA^{-1}x) > 0$. Conversely, if g has only positive leading principal minors, then (5.3) enables us to construct the necessary factorization. ∎

We now consider the dynamical system (4.6) associated with this factorization. Firstly we identify L^* with $L(G)$ via the nondegenerate bilinear

Ad-invariant form $\langle\langle\ ,\ \rangle\rangle$ on $L(G)$:

$$\langle\langle A, B\rangle\rangle = \mathrm{Tr}(AB), \qquad A, B \in L(G).$$

Let φ be the Casimir function [see (4.5)] on $L(G)$:

$$\varphi(A) = \tfrac{1}{2}\langle\langle A, A\rangle\rangle.$$

Clearly $\nabla\varphi(A) = A$, and we arrive at the following dynamical system [see (4.6a), (5.1)]:

$$\dot{A} = \left[A, \frac{\pi_+ - \pi_-}{2} A\right] = \left[A, \left(\pi_+ + \frac{\pi_0}{2}\right)A\right], \qquad A(0) = A_0. \quad (5.4)$$

Let

$$\exp(A_0 t) = \rho_+(t)\rho_-(t), \qquad \pi_0(\rho_+(t)) = \pi_0(\rho_-(t)) \quad (5.5)$$

be the factorization (5.2). Then by (4.8) we have for the solution to (5.4)

$$A(t) = \rho_+(t)^{-1}A(0)\rho_+(t).$$

Consider the case $A_0 = A_0^T$. From (5.5) we obtain

$$\exp(A_0 t) = \rho_-(t)^T\rho_+(t)^T, \qquad \pi_0\left(\rho_-(t)^T\right) = \pi_0\left(\rho_+(t)^T\right).$$

This is another factorization (5.2), but by Theorem 3 there exists only one such factorization. Hence $\rho_-(t)^T = \rho_+(t)$, and (5.5) takes the form

$$\exp(A_0 t) = \rho_+(t)\rho_+(t)^T. \quad (5.6)$$

Moreover, the QR-type algorithm associated with the factorization (5.6) (with a positive definite symmetric initial value) is exactly the well-known Cholesky algorithm for finding the eigenvalues of a positive definite matrix (see e.g. [20]). Let $g_0 \doteq \exp(A_0)$, $A_0 = A_0^T$, and $g_1, g_2, \dots$ be the iterations of the Cholesky algorithm. Then by Theorem 5 we have

$$g_k = \exp(A(k)), \qquad k = 0, 1, \dots,$$

where $A(t)$ is a solution to (5.4).

This example shows that there are some interesting factorizations associated with solutions to the Yang-Baxter equation which are beyond the AKS case.

We now consider two factorizations in the real symplectic group which are relevant to QR-type algorithms for the solution of the algebraic Riccati equation of control theory. We don't discuss here any aspects of the practical implementation of these algorithms (see also [4]).

Let us define the $2n$-by-$2n$ real matrix J by

$$J=\begin{bmatrix} 0 & E_n \\ -E_n & 0 \end{bmatrix},$$

where E_n is n-by-n identity matrix. The real symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ is the set of $S \in \mathrm{GL}(2n,\mathbb{R})$ such that $S^TJS = J$. The associated Lie algebra is the set of $2n$-by-$2n$ matrices H such that $(JH)^T = JH$. These matrices are called Hamiltonian. A $2n$-by-$2n$ matrix H is Hamiltonian if and only if it has the form

$$\begin{bmatrix} A & L \\ Q & -A^T \end{bmatrix},$$

where $L^T = L$, $Q^T = Q$. An n-dimensional vector subspace $N \subset \mathbb{R}^{2n}$ is called Lagrangian if, given $x, y \in N$, we have $x^TJy = 0$. We denote by $(e_1,\dots,e_n, f_1,\dots,f_n)$ the standard basis in $\mathbb{R}^{2n}$, and by L_e (L_f) the linear span of $e_1,\dots,e_n$ ($f_1,\dots,f_n$). Clearly L_e and L_f are Lagrangian subspaces in $\mathbb{R}^{2n}$.

EXAMPLE 5. Let G_+ be the Lie subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ such that

$$G_+ = \left\{ S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \in \mathrm{Sp}(2n,\mathbb{R}) : S_{12} = 0,\ S_{11} = S_{22} = E \right\}.$$

As is easily seen,

$$S = \begin{bmatrix} E & 0 \\ K & E \end{bmatrix} \in G_+$$

if and only if $K^T = K$. Let G_- be the Lie subgroup of $\mathrm{Sp}(2n, \mathbb{R})$:

$$G_- = \left\{ S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \in \mathrm{Sp}(2n, \mathbb{R}) : S_{21} = 0 \right\}.$$

Clearly the corresponding Lie algebras $L(G_{\pm})$ are

$$L(G_+) = \left\{ \begin{bmatrix} 0 & 0 \\ Q & 0 \end{bmatrix} : Q^T = Q \right\}, \qquad L(G_-) = \left\{ \begin{bmatrix} A & L \\ 0 & -A^T \end{bmatrix} : L^T = L \right\}.$$

We have

$$L(G_+) \oplus L(G_-) = L(\mathrm{Sp}(2n, \mathbb{R})) \triangleq \mathrm{Sp}(2n, \mathbb{R}).$$

Further, $G_+ \cap G_- = e$ and $S \in G_+ G_-$ if and only if $S \in \mathrm{Sp}(2n, \mathbb{R})$ and $\det S_{11} \neq 0$. Given $S \in \mathrm{Sp}(2n, \mathbb{R})$,

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \qquad \det S_{11} \neq 0,$$

we have

$$S = \begin{bmatrix} E & 0 \\ K & E \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ 0 & (S_{11}^{-1})^T \end{bmatrix}, \qquad K = S_{12} S_{11}^{-1}. \tag{5.7}$$

For

$$H = \begin{bmatrix} A & L \\ Q & -A^T \end{bmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$$

consider the factorization of $S(t) = \exp(Ht)$

$$\begin{aligned} S(t) &= \begin{bmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{bmatrix} \\ &= \begin{bmatrix} E & 0 \\ K(t) & E \end{bmatrix} \begin{bmatrix} S_{11}(t) & S_{12}(t) \\ 0 & [S_{11}(t)^{-1}]^T \end{bmatrix}. \end{aligned}$$

By (5.7) we obtain

$$\dot{K}(t) = Q - A^T K(t) - K(t)A - K(t)LK(t), \tag{5.8}$$

$K(0) = 0$, i.e., $K(t)$ satisfies the matrix Riccati equation of control theory. We further consider the dynamical system [see (4.6a)] associated with the solution $R = \pi_+ - \pi_-$ of the Yang-Baxter equation. Here $\pi_\pm$ is the projection of $L(G)$ onto $L(G_\pm)$ along $L(G_\mp)$:

$$\dot{H} = [H, \pi_+(H)], \qquad H(0) = \begin{bmatrix} A & L \\ Q & -A^T \end{bmatrix}. \tag{5.9}$$

By (4.8) we have

$$\begin{aligned} H(t) &= \begin{bmatrix} E & 0 \\ -K(t) & E \end{bmatrix} H(0) \begin{bmatrix} E & 0 \\ K(t) & E \end{bmatrix} \\ &= \begin{bmatrix} A + LK(t) & L \\ \mathscr{R}(K(t)) & -(A + LK(t))^T \end{bmatrix}, \end{aligned} \tag{5.10}$$

where $\mathscr{R}(K) = Q - A^T K - KA - KLK$.

Let $g_0 = \exp(H(0))$ and $g_1, g_2, \dots$ be the iterations of the QR-type algorithm associated with the factorization (5.7). Then by Theorem 5 we have

$$g_k = \exp(H(k)), \qquad k = 0, 1, \dots, \tag{5.11}$$

where $H(t)$ is the solution (5.10) to (5.9). Moreover, as an easy computation shows, if

$$g_k = \begin{bmatrix} \Phi_{11}(k) & \Phi_{12}(k) \\ \Phi_{21}(k) & \Phi_{22}(k) \end{bmatrix} = \begin{bmatrix} E & 0 \\ T(k) & E \end{bmatrix} \begin{bmatrix} \Phi_{11}(k) & \Phi_{12}(k) \\ 0 & [\Phi_{11}(k)^{-1}]^T \end{bmatrix},$$

$k = 0, 1, 2, \dots,$ then

$$\Phi_{12}(k+1) = \Phi_{12}(k), \qquad \Phi_{11}(k+1) = \Phi_{11}(k) + \Phi_{12}(0)T(k), \tag{5.12}$$

with

$$T(k+1) = [\Phi_{11}(k)^{-1}]^T T(k) [\Phi_{11}(k+1)]^{-1}, \qquad K(k+1) = K(k) + T(k),$$

where $K(t)$ is the solution to (5.8) with $K(0)=0$. We now study the asymptotic behavior of (5.12) at $k \to \infty$. Let the spectrum σ of the matrix $H(0)$ possess the following property:

$$\sigma \cap i\mathbb{R} = \varnothing, \tag{5.13}$$

where $i\mathbb{R}$ is the imaginary axis of the complex plane. Then $\sigma = \sigma_+ \cup \sigma_-$, $\sigma_+ \cap \sigma_- = \varnothing$, $\sigma_+ = \{\lambda \in \sigma : \operatorname{Re}\lambda > 0\}$, $\sigma_- = \sigma \backslash \sigma_+$.

Consider the corresponding decomposition

$$\mathbb{R}^{2n} = L_+ \oplus L_-$$

into the direct sum of $H(0)$-invariant subspaces. It turns out (see e.g. [8]) that $L_\pm$ are Lagrangian subspaces in $\mathbb{R}^{2n}$. It is well known (see e.g. [8]) that the set $\mathscr{L}(n)$ of all Lagrangian subspaces in $\mathbb{R}^{2n}$ is endowed naturally with the structure of a smooth manifold, the real symplectic group acts smoothly and transitively on $\mathscr{L}(n)$, and

$$\mathscr{L}(n) \approx \operatorname{Sp}(2n, \mathbb{R})/G_-. \tag{5.14}$$

Further, under the assumptions (5.13) and

$$L_e \cap L_- = \varnothing \tag{5.15}$$

we have (see e.g. [16])

$$\lim \exp(H(0)t) \cdot L_e = L_+, \qquad t \to +\infty, \tag{5.16}$$

in $\mathscr{L}(n)$. If $L_+ \subset G_+ G_- L_e$ (which is equivalent to $L_+ \cap L_f = \varnothing$), then by Proposition 2 (taking into account $G_+ \cap G_- = e$)

$$\lim \begin{bmatrix} E & 0 \\ K(t) & E \end{bmatrix} = \begin{bmatrix} E & 0 \\ K_+ & E \end{bmatrix}, \qquad t \to +\infty, \tag{5.17}$$

where $K_+ = K_+^T$ is a unique matrix such that

$$\begin{bmatrix} E & 0 \\ K_+ & E \end{bmatrix} L_e = L_+,$$

i.e., L_+ is the linear span of the columns of the matrix

$$\begin{bmatrix} E \\ K_+ \end{bmatrix}.$$

By (5.17),(5.10) we obtain

$$\lim_{t \to +\infty} H(t) = \begin{bmatrix} A + LK_+ & L \\ 0 & -(A + LK_+)^T \end{bmatrix} = H_+,$$

whence by (5.11) $\lim g_k = \exp(H_+)$, $k \to +\infty$. Moreover, (5.17) implies that K_+ is the antistabilizing solution to the algebraic Riccati equation

$$\mathscr{R}(K) = Q - A^T K - KA - KLK = 0. \tag{5.18}$$

Thus, the iterative procedure (5.12) yields a method for the solution of the algebraic Riccati equation (5.18) of control theory.

REMARK. We have (see [18]) the complete information about the phase portrait of the Riccati differential equation (5.8). This enables us to study the asymptotic behavior of solutions to (5.9) and consequently via (5.11) of the QR-type algorithm (5.12) without the assumptions (5.13),(5.15).

Let $(\mathscr{L}, [\ ,\ ])$ be a Lie algebra, and $\varepsilon: \mathscr{L} \to \mathscr{L}$ be a Lie-algebra isomorphism such that $\varepsilon^2 = E_{\mathscr{L}}$, where $E_{\mathscr{L}}$ is an identity operator on $\mathscr{L}$. Further, let $\mathscr{L}_0, \mathscr{L}_\pm$ be Lie subalgebras of $\mathscr{L}$ such that

$$[\mathscr{L}_0, \mathscr{L}_\pm] \subset \mathscr{L}_\pm, \qquad \mathscr{L} = \mathscr{L}_- \oplus \mathscr{L}_0 \oplus \mathscr{L}_+, \tag{5.19}$$

as a direct sum of vector subspaces,

$$\varepsilon(\mathscr{L}_-) = \mathscr{L}_+, \qquad \mathscr{L}_0 \subset P = \{x \in \mathscr{L} : \varepsilon(x) = -x\}. \tag{5.20}$$

We introduce the Lie subalgebra $\mathscr{K}$ of $\mathscr{L}$:

$$\mathscr{K} = \{x \in \mathscr{L} : \varepsilon(x) = x\}. \tag{5.21}$$

First of all we have

$$\mathscr{L} = \mathscr{K} \oplus \mathscr{L}_0 \oplus \mathscr{L}_+. \tag{5.22}$$

Indeed, given $x \in \mathscr{L}$, denote the decomposition x according to (5.19) by

$$x = x_- + x_0 + x_+. \tag{5.23}$$

We have

$$x = [x_- + \varepsilon(x_-)] + x_0 + [x_+ - \varepsilon(x_-)],$$

where $x_- + \varepsilon(x_-) \in \mathscr{K}$ by (5.21), $x_+ - \varepsilon(x_-) \in \mathscr{L}_+$ by (5.20). If $z \in \mathscr{K} \cap (\mathscr{L}_0 \oplus \mathscr{L}_+)$, then $z = z_0 + z_+$, $\varepsilon(z) = -z_0 + \varepsilon(z_0)$, $z = \varepsilon(z)$ by (5.20), (5.21). Thus, $z_0 = z_+ = 0$, i.e. $z = 0$.

Let $M \subset \mathscr{L}_-$ be such that

$$[\mathscr{L}_0, M] \subset M, \qquad [\mathscr{L}_+, M] \subset M \oplus \mathscr{L}_0 \oplus \mathscr{L}_+. \tag{5.24}$$

Consider the dynamical system on $\mathscr{L}$

$$x = [x, \pi_{\mathscr{K}}(x)], \tag{5.25}$$

where $\pi_{\mathscr{K}}: \mathscr{L} \to \mathscr{K}$ is the projection onto K along $\mathscr{L}_0 \oplus \mathscr{L}_+$ [see (5.22)].

PROPOSITION 12. *Under the assumptions* (5.24) *the vector subspace* $V = M \oplus \mathscr{L}_0 \oplus \mathscr{L}_+$ *is an invariant manifold for the dynamical system* (5.25).

Proof. We have $\pi_{\mathscr{K}}(x) = x_- + \varepsilon(x_-)$. Given $x \in V$, we have $[x, \pi_{\mathscr{K}}(x)] = [x_-, \varepsilon(x_-)] + [x_0, x_-] + [x_0, \varepsilon(x_-)] + [x_+, x_-] + [x_+, \varepsilon(x_-)] \in V$ by (5.24), (5.19). ■

EXAMPLE 6. Let $\mathscr{L} = \mathrm{Sp}(2n, \mathbb{R})$, $\mathscr{L}_0$ be the set of diagonal matrices in $\mathrm{Sp}(2n, \mathbb{R})$,

$$\mathscr{L}_+ = \left\{ \begin{bmatrix} A & L \\ 0 & -A^T \end{bmatrix} \in \mathrm{Sp}(2n, \mathbb{R}) : A \text{ is strictly upper triangular} \right\},$$

$$\mathscr{L}_- = \left\{ \begin{bmatrix} A & 0 \\ Q & -A^T \end{bmatrix} \in \mathrm{Sp}(2n, \mathbb{R}) : A \text{ is strictly lower triangular} \right\}.$$

$$\varepsilon(x) = -x^T, \qquad x \in \mathrm{Sp}(2n, \mathbb{R}).$$

Clearly, the conditions (5.19), (5.20) hold, and $\mathscr{K}$ is the set of skew-symmetric Hamiltonian matrices. Consider the set $M \subset \mathscr{L}^-$ given by

$$M = \left\{ \begin{bmatrix} A & 0 \\ Q & -A^T \end{bmatrix} \in \mathscr{L}_- : A = \|a_{ij}\|,\ a_{ij} = 0 \text{ unless } j+1 = i, \right.$$

$$\left. Q = \|q_{ij}\|,\ q_{ij} = 0 \text{ unless } (i,j) = (n,n) \right\}.$$

It is easily verified that (5.24) holds and hence $M \oplus \mathscr{L}_0 \oplus \mathscr{L}_+$ is an invariant manifold for (5.25). Let us remark that $M \oplus \mathscr{L}_0 \oplus \mathscr{L}_+$ is exactly the set of $2n$-by-$2n$ matrices in Hamiltonian-Hessenberg form as it was introduced by R. Byers [5]. Moreover, the factorization of the real symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ corresponding to the decomposition (5.22) is exactly the symplectic QR factorization considered in [5]. According to [5], the corresponding QR algorithm (applied to Hamiltonian-Hessenberg reductions) seems to be an efficient way for the solution of the algebraic Riccati equation.

A Lie-theoretic description of Hessenberg and Hamiltonian-Hessenberg forms in terms of root spaces and Iwasawa decompositions is also given in [24]. We can consider this example in a more general setting. Let $\mathscr{L}$ be a splittable real semisimple noncompact Lie algebra, ε be a Cartan involution, $\mathscr{L}_0$ be a Cartan subalgebra. We take $\mathscr{L}_0 \oplus \mathscr{L}_\pm$ to be a positive (negative) Borel subalgebra in $\mathscr{L}$. Let M be the direct sum of root subspaces corresponding to the simple negative roots. Then (5.24) holds. We say that the elements of the vector subspace $V = M \oplus \mathscr{L}_0 \oplus \mathscr{L}_+$ are in the abstract upper Hessenberg form. In the case $\mathscr{L} = \mathrm{sl}(n, \mathbb{R})$ we arrive at the usual upper Hessenberg matrices, and for $\mathscr{L} = \mathrm{Sp}(2n, \mathbb{R})$ we obtain Hamiltonian-Hessenberg matrices. Let us remark that (5.22) is the usual Iwasawa decomposition for $\mathscr{L}$ (see e.g. [11]) in this context. We think that the abstract Hessenberg form may well be of the same importance as the usual Hessenberg form for the practical implementation of general QR-type algorithms.

Terminological remark. The algebraic Riccati equation is known in the Soviet control literature under the name Lurie equation. It was A. I. Lurie [12] who introduced this equation in control theory. V. A. Yackubovich, who proposed this name, developed some frequency-domain methods (see [22] and [23] for further references) for the solution of this equation. These results remained mostly unknown in the West (except for the well-known Kalman-Yackubovich lemma), partially due to the abovementioned difference in the terminology.

6. CONCLUDING REMARKS

In this paper we considered the linkage between QR-type algorithms and solutions to the Yang-Baxter equation. We have shown that every solution to the Yang-Baxter equation corresponds with the QR-type algorithm and the dynamical system on the Lie coalgebra provided we have a sequence P_i, $i = 0, 1, 2 \dots$, satisfying (2.2). The relationship between the QR-type algorithm and the dynamical system is also established (see Theorem 5). We described the structure of solutions of the Yang-Baxter equation and considered two examples relevant to the algebraic Riccati equation of control theory. Our approach enables one to take into consideration shifted and generalized QR-type algorithms on the level of associated dynamical systems. We have not described this situation, because that has been already done by D. Watkins and L. Elsner [21] for the AKS matrix case.

The Yang-Baxter equation appeared in the form that we employed here in [15]. M. Semenov-Tian-Shansky introduced it in connection with completely integrable systems. For this latter topic the infinite-dimensional case is the main interest (e.g. the Yang-Baxter equation on affine Lie algebras). It is a remarkable fact that the same is true for possible system-theoretic applications. Indeed, the factorizations in the Lie algebra of transfer matrix functions are of the same nature as in the case of completely integrable systems. We plan to pursue this topic separately.

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